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LETTER TO THE EDITOR

Eigenfunctions for SU(ν) particles with $1/r^2$ interaction in harmonic confinement

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Abstract. We find a set of exact eigenfunctions which provide the energy spectrum for the quantum N-body Calogero–Sutherland model for fermions or bosons with SU(ν) spin degrees of freedom moving in a harmonic confinement potential. The eigenfunctions are explicitly constructed as a simple product of the Jastrow wavefunction for the ground state and the Hermite polynomials introduced to generate the excited states. The corresponding energy spectrum is given by a sum of the correlated ground-state energy and of the excited-state energy for SU(ν) free particles in a harmonic well.

Integrable systems with inverse-square ($1/r^2$) interaction in one dimension were introduced by Calogero [1] and Sutherland [2], and more recently extended to the spin chain by Haldane [3] and Shastry [4]. Models with periodic boundary conditions have then been studied intensively and the corresponding wavefunctions have been constructed explicitly [5–13].

For the $1/r^2$ systems with harmonic confinement potential [2, 10, 14–17], the formal algebraic structure has been clarified recently via the construction of appropriate annihilation and creation operators [10, 14], which enables one to prove the integrability for a class of the SU(ν) models with confinement [16]. Such an operator formalism is elegant, giving an answer about the spectrum and the degeneracy of the energy levels. However, we still lack expressions for the eigenfunctions of excited states for the $1/r^2$ system with harmonic confinement. The systematic construction of eigenfunctions should also provide a microscopic foundation of the renormalized-harmonic oscillator hypothesis [18] which is based on a variant of the asymptotic Bethe ansatz method [2, 3, 8].

Several authors attempted to construct explicitly the wavefunctions for the excited states of the Calogero–Sutherland model with confinement, but this problem has not been solved generally even for the single-component case. Calogero [1] obtained early results for the eigenfunctions of the system containing $N = 3$ and $N = 4$ particles. Calogero's attempt to obtain the solution for the N-body system has left open the problem of finding an explicit expression for the eigenfunctions: a systematic construction of the polynomial solutions to the generalized Laplace equations has not been accomplished yet [1]. A special solution for $N = 5$ has been presented by Gambardella [19]. The formal approach using the operator algebra [10, 14] yields expressions for wavefunctions that are simple only for a small number of particles in the system. It has indeed been claimed [14] that the expressions quickly become cumbersome due to complicated sums in the definition of operators. Similar difficulties may happen if the wavefunctions for particles with internal degrees of freedom are constructed from the operators introduced by Minahan and Polychronakos [16].

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The purpose of this letter is to construct explicitly a set of the *exact* eigenfunctions for the *N-body* Calogero–Sutherland model where the particles with SU(ν) internal degrees of freedom move in an external harmonic confining potential. It turns out that the eigenfunctions obtained here cover the *full energy spectrum*, although they belong to a special set of complete eigenfunctions. To illustrate this approach, we start with the construction of wavefunctions for the single-component case. We then turn to the case of particles with the internal SU(ν) spin degrees of freedom, and present the eigenfunctions and the corresponding energy spectrum.

Let us introduce the Hamiltonian in units of $\hbar^2/m$ for the inverse-square model of spinless particles confined by a harmonic potential $\frac{1}{2}m\omega_0^2x^2$ [1, 2],

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{m^2\omega_0^2}{2\hbar^2}\sum_{i=1}^{N}x_i^2 + \sum_{j>i}\frac{\lambda(\lambda \mp 1)}{(x_j - x_i)^2} \tag{1}$$

where N is the total number of particles and $\lambda \geqslant 0$ is a dimensionless constant of pair coupling. The upper (lower) sign in the interaction term holds for bosons (fermions).

We first recall a characteristic feature of the wavefunction inherent in the $1/r^2$ models: the ground-state wavefunction is of the Jastrow type, expressed as a product of the Jastrow factor and Gaussian functions [2]. It has been known that the construction of this wavefunction exhibits a remarkable similarity to that of Laughlin's wavefunction for the fractional quantum Hall (FQH) states with the filling $1/p$, $\Psi_p = z^p \times$ (Gaussians), where z is a Vandermonde determinant [20]. In the FQH states, it is further known that the wavefunctions for excited states are obtained by multiplying an appropriate polynomial to the ground-state wavefunction [20]. One naturally expects that even for excited states, the construction of eigenfunctions for the $1/r^2$ models can be quite analogous to that for the FQH states. It was actually demonstrated that this is indeed the case for the systems with periodic boundary conditions [2–4, 7, 9].

Based on these observations, we thus propose the following set of Jastrow-type ansatz eigenfunctions for the system of bosons and fermions described by the above Hamiltonian (1),

$$\Psi(x_1, x_2, \ldots, x_N) = |z|^\lambda z^\gamma \Phi(x_1, x_2, \ldots, x_N) \tag{2}$$

where $z = \prod_{j>i}(x_j - x_i)$ is the Vandermonde determinantal product, $\gamma = 0$ ($\gamma = 1$) for bosons (fermions), and Φ is assumed to be a symmetric polynomial. The second factor z^γ is introduced to generate the symmetry (antisymmetry) property of the wavefunction for the boson (fermion) case. Note that $\Psi(\lambda = 0)$ corresponds to a solution of the non-interacting case. Motivated by the analogy to the FQH states, let us further assume that Φ is a product of a polynomial F and a function G, namely $\Phi = FG$. The function G is a product of Gaussians,

$$G(x_1, \ldots, x_N) = \prod_{i=1}^{N}\exp\left(-\frac{m\omega_0}{2\hbar}x_i^2\right) \tag{3}$$

which can naturally take into account the effects of the harmonic confinement. The polynomial F introduced here,

$$F(x_1, \ldots, x_N) = \sum_{m_1+\cdots+m_N=I} a_{m_1\ldots m_N}\prod_{i=1}^{N}H_{m_i}\left(\sqrt{\frac{m\omega_0}{\hbar}}x_i\right) \tag{4}$$

is a linear combination of Hermite polynomials H_{m_i} with unknown amplitudes $a_{m_1\ldots m_N}$ which should satisfy the condition of symmetry, $a_{m_1\ldots m_k\ldots m_\ell\ldots m_N} = a_{m_1\ldots m_\ell\ldots m_k\ldots m_N}$. Note that the excitations are labelled by the quantum numbers in the polynomial F, $m_i = 0, 1, 2\ldots$ $(i = 1, 2, \ldots, N)$. The sum in the definition of the polynomial F is taken over quantum numbers m_i $(i = 1, 2, \ldots, N)$ which satisfy $\sum_{i=1}^{N} m_i = I$, where I is a given non-negative integer.

We show that ansatz wavefunction (2) with (3) and (4) satisfies the Schrödinger equation $H\Psi = E\Psi$ in the sector $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$. The proof can easily be extended to the whole configuration space. The application of the Hamiltonian (1) on the ansatz eigenfunction (2) results in the following expression,

$$\frac{1}{\Psi}H\Psi = \frac{m\omega_0}{\hbar}\left[\frac{1}{2}(\lambda+\gamma)N(N-1)+\frac{N}{2}+I\right] - \frac{\lambda+\gamma}{F}\sum_{\ell>k}\frac{1}{x_\ell - x_k}\left(\frac{\partial F}{\partial x_\ell} - \frac{\partial F}{\partial x_k}\right). \quad (5)$$

We would like to eliminate the cross-term containing the derivatives of the function F. Such a requirement leads to the additional condition imposed on the amplitudes $a_{m_1\ldots m_N}$ for any $k < \ell$, which is satisfied when the polynomial F contains two terms with the amplitudes related by

$$m_k a_{m_1\ldots m_k\ldots m_\ell\ldots m_N} = (m_\ell + 1)a_{m_1\ldots m_k-1\ldots m_\ell+1\ldots m_N} \quad (6)$$

for any $m_k > 0$ and $m_\ell < I$. It is further required that the sum in the definition of the polynomial F is taken over *all* possible combinations of integers m_i satisfying $\sum_{i=1}^{N} m_i = I$. The conditional equation (6) is solved by the factorization of amplitudes $a_{m_1\ldots m_N}$ (with $0! = 1$),

$$a_{m_1\ldots m_N} = \prod_{i=1}^{N}\frac{1}{m_i!}. \quad (7)$$

Consequently, the eigenfunction for the Calogero–Sutherland-type model (1) of spinless particles moving in a harmonic potential can be written as

$$\Psi = \prod_{j>i}[|x_j - x_i|^\lambda (x_j - x_i)^\gamma] \sum_{m_1+\ldots+m_N=I}\prod_{i=1}^{N}\frac{1}{m_i!}H_{m_i}\left(\sqrt{\frac{m\omega_0}{\hbar}}x_i\right)\exp\left(-\frac{m\omega_0}{2\hbar}x_i^2\right). \quad (8)$$

The corresponding energy spectrum follows from (5),

$$E(N; I) = \tfrac{1}{2}\hbar\omega_0[(\lambda+\gamma)N(N-1) + N] + \hbar\omega_0 I \quad (9)$$

where a non-negative integer I labels the excitation. The ground-state wavefunction and the corresponding eigen-energy (obtained for $I = 0$) reproduce the known exact results [2]. Note that the case of non-interacting particles (both fermions and bosons) is achieved by $\lambda = 0$.

The energy spectrum (9) is a sum of the N-particle correlated ground-state energy and the spectrum of N non-interacting fermions or bosons in a harmonic well, as should be expected [14]. For example, the excitation energy of the particle–hole type, which is labelled by a positive integer I with fixed N, is independent of the interaction strength λ. Although the set of eigenfunctions (8) provides us with a special series of the eigenfunctions for N particles, it supplies *all* energy levels of the system, as can be seen when one compares the

energy spectrum (9) with the results deduced by other techniques [14]. Hence, the energy levels obtained from the wavefunction (8) cover all excitation energies. It is, however, still open to count the degeneracy of each level within the present approach. We note that one can employ other methods [14] for counting the degeneracy.

We now generalize the model for particles with SU(ν) spin degrees of freedom. The integrability and the ground-state wavefunction (up to $N = 6$ particles) were shown for the model by Minahan and Polychronakos [16]. Recently, we have constructed the ground-state wavefunction for an arbitrary number of particles [17]. Using the approach outlined above, let us now obtain a set of excited-state wavefunctions and the energy spectrum for particles with SU(ν) internal degrees of freedom. The SU(ν) integrable generalization of the Hamiltonian in units of $\hbar^2/m$ takes the form [7, 10, 12]

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{m^2\omega_0^2}{2\hbar^2}\sum_{i=1}^{N}x_i^2 + \sum_{j>i}\frac{\lambda(\lambda + P_{ij}^{\sigma})}{(x_j - x_i)^2} \tag{10}$$

where the spin-exchange operator P_{ij}^{σ} of particles i, j has been introduced. We propose the Jastrow-type ansatz eigenfunction, namely the trial function reads

$$\Psi(x_1\sigma_1, \ldots, x_N\sigma_N) = \left\{\prod_{j>i}|x_j - x_i|^{\lambda+\gamma-1}(x_j - x_i)^{\delta\sigma_j\sigma_i-\gamma+1}\exp\left[\mathrm{i}\frac{\pi}{2}\,\mathrm{sgn}(\sigma_j - \sigma_i)\right]\right\}$$
$$\times\left[\sum_{m_1+\cdots+m_N=I}\prod_{i=1}^{N}\frac{1}{m_i!}H_{m_i}\left(\sqrt{\frac{m\omega_0}{\hbar}}x_i\right)\right]\prod_{i=1}^{N}\exp\left(-\frac{m\omega_0}{2\hbar}x_i^2\right) \tag{11}$$

where the indices σ_i ($i = 1, 2, \ldots, N$) denote the SU(ν) spin of each particle. The exponent γ is equal to 0 and 1 for bosons and fermions, respectively. The third factor is introduced to take into account the symmetry of SU(ν) degrees of freedom.

Here we make a brief comment on the wavefunction (11). As in other models with $1/r^2$ interaction [2, 7], for instance in the fermion case, the eigenfunction $\Psi(\lambda = 0, \gamma = 1)$ is a solution for non-interacting SU(ν) fermions in a harmonic well. So, the wavefunction (11) can be rewritten as $\Psi = \prod_{j>i}|x_j - x_i|^{\lambda}\Psi(\lambda = 0, \gamma = 1)$. One can clearly see, from this expression, the analogy to Jain's construction [21] of the wavefunction for hierarchical FQH states in which the Jastrow factor is introduced to non-interacting electrons of filled Landau levels with the filling p. Furthermore, following the proof for the SU(2) case [17], it is easily checked that the eigenfunction Ψ is a product written as $\Psi = F\Psi_G$, where F is defined by (4) and (7), and Ψ_G is the ground-state wavefunction of the Hamiltonian (10) for a given spin configuration. This decomposition implies that we are now looking for the eigenfunctions whose symmetry is the same as Ψ_G, since a completely symmetric polynomial F does not change the symmetry property.

It is now straightforward to show that the substitution of the trial function (11) into the Schrödinger equation with the Hamiltonian (10) yields the expression for the eigen-energy

$$E(N_1, N_2, \ldots, N_\nu; I) = \tfrac{1}{2}\hbar\omega_0\left[\lambda N(N-1) + \sum_{\alpha=1}^{\nu}N_\alpha^2\right] + \hbar\omega_0 I \tag{12}$$

where the configuration of spins is denoted by the number of particles, $N_1, N_2, \ldots, N_\nu$ ($\sum_{\alpha=1}^{\nu}N_\alpha = N$). The trial function (11) is therefore an eigenfunction of the Hamiltonian (10). The correlations via $1/r^2$ interaction appear only in the ground-state energy for a given

spin configuration, $E_G = \frac{1}{2}\hbar\omega_0[\lambda N(N-1) + \sum_{\alpha=1}^{\nu} N_\alpha^2]$. The excitations do not include any effects of interactions, provided that the number of electrons, N_α, is kept fixed [18]. The level of the spin-independent particle–hole excitations is determined by the quantum number $I = \sum_{i=1}^{N} m_i$; the lowest-lying excitation is obtained for $I = 1$, the next one for $I = 2$, etc. The case of non-interacting fermions is achieved for $\lambda = 0$, whereas for bosons the case of $\lambda = 0$ corresponds to particles with an infinite hard core, as discussed in [2] and [6]. Notice that λ in our notation for the multicomponent model (10) corresponds to $\lambda - 1$ in the notation of [2] and [6].

The spectrum of the Hamiltonian (1) or (10) is the same as the spectrum of non-interacting particles shifted only by the correlated ground-state energy obtained previously [17]. Comparing with the spectrum suggested in the hypothesis [18], one can see that the energies of all quantum levels are produced directly from wavefunctions (11). As is the case for the other $1/r^2$ models [3, 13, 22], the degeneracy of each energy level is expected to be given by that of non-interacting SU(ν) particles, described by independent sets of quantum numbers m_i, $i = 1, 2, \ldots, N$. In order to figure out the problem of degeneracy and obtain the corresponding wavefunctions microscopically, it is desirable to analyse symmetry properties of the model in detail [14, 16]. In particular, we think that Yangian symmetry discussed in [22] may play a key role for the classification of the complete spectrum of the present SU(ν) model.

In conclusion, we have presented a set of the excited-state wavefunctions for SU(ν) generalization of the quantum Calogero–Sutherland Hamiltonian with harmonic confinement. Although the present wavefunctions belong to a special subset of complete eigenfunctions, it is remarkable that the constructed set of eigenfunctions provides us exactly with all the energy levels for the excited states.

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